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THE SIZE OF SUMS OF SETS, II

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ABSTRACT

We prove inequalities which give lower bounds for the Lebesgue measures of sets $E+K$ where K is a certain kind of Cantor set. For example, if C is the Cantor middle-thirds subset of the circle group $\mathbf{T}$, then

$$m(E)^{1-\log 2/\log 3} \leqq m(E+C)$$

for every Borel $E \subseteq \mathbf{T}$.

Two of the attributes of a locally compact abelian group G are its Haar measure m and its addition operation. An aspect of the relationship between these is the behavior of the Haar measure of sets which are sums. The objects of study in [3] were certain inequalities of the form

$$\delta m(E)^{\alpha} \leqq m(E+K) \tag{1}$$

holding for some measurable $K \subseteq G$ such that $m(K)=0$, some $\delta>0$ and $\alpha \in (0,1)$ depending on K, and all measurable $E \subseteq G$ for which $E+K=\{e+k : e \in E, k \in K\}$ is measurable. The particular instances of (1) which furnished the motivation for [3] had $G=\mathbf{R}^n$ and K a suitable k-dimensional surface in $\mathbf{R}^n$ $(1 \leqq k < n)$. Here we consider inequalities (1) which hold when G is the circle group $\mathbf{T}$ and when K is a Cantor-like set. This paper is organized as follows. In §1 some conditions are given which are equivalent to the existence of an inequality (1). These have a certain intrinsic interest, and one of them is necessary for our later work. The result of §2 is a sufficient condition for certain generalized Cantor sets K to satisfy (1). It follows easily from this theorem that if C is the Cantor middle-thirds set, then

$$m(E)^{1-(\log 2/\log 3)} \leqq m(E+C) \tag{2}$$

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for every Borel $E \subseteq \mathbf{T}$. In §3 we establish inequalities similar to (2) but with C replaced by certain of its subsets. We also comment on an interesting related inequality of Brown and Moran.

§1. Let G be a locally compact abelian group with Haar measure m. Denote by C_c^+ the space of nonnegative continuous functions of compact support on G. For $f \in C_c^+$ and $p \in (0,\infty)$, let

$$\|f\|_p = \left(\int_G |f|^p dm\right)^{1/p}.$$

THEOREM 1. *Suppose K is a compact subset of G and $0 < \alpha < 1$. The following are equivalent*:

(a) *there is $\delta > 0$ such that $\delta m(E)^\alpha \leqq m(E+K)$ for all compact $E \subseteq G$;*

(b) *there is $M > 0$ such that $m(\bigcap_{k\in K}(F-k)) \leqq Mm(F)^{1/\alpha}$ for all compact $F \subseteq G$;*

(c) *there is $\delta > 0$ such that for all $f \in C_c^+$*

$$\delta \|f\|_{1/\alpha} \leqq \int_G \sup\{f(x-k): k \in K\} dm(x);$$

(d) *there is $M > 0$ such that for all $g \in C_c^+$*

$$\int_G \inf\{g(x+k): k \in K\} dm(x) \leqq M\|g\|_\alpha.$$

PROOF. To prove that (a) implies (b), let $E = \bigcap_{k\in K}(F-k)$. Then $E + K \subseteq F$, so if δ is as in (a) we can take $M = \delta^{-1/\alpha}$ in (b). The proof that (b) implies (a) is similar.

To prove that (c) implies (d), let $f(x) = \inf\{g(x+k)^\alpha : k \in K\}$. Then $f \in C_c^+$ because of the uniform continuity of g and the compactness of K, and $\sup\{f(x-k): k \in K\} \leqq g(x)^\alpha$. Thus if δ is as in (c) we can take $M = \delta^{-1/\alpha}$ in (d). The proof that (d) implies (c) is similar.

That (c) implies (a) would be obvious if we could take f in (c) to be the indicator function χ_E. It can be proved by approximating χ_E with $f \in C_c^+$.

To show that (a) implies (c), put $h(x) = \sup\{f(x-k): k \in K\}$. Then for $s > 0$ we have

$$\{f > s\} + K \subseteq \{h > s\}.$$

Write $\lambda(f,s) = m\{f > s\}$ and define $\lambda(h,s)$ similarly. Since (a) must hold also when E is open,

$$\delta\lambda(f,s)^\alpha \leqq \lambda(h,s).$$

Now

$$\begin{aligned}\|f\|_{1/\alpha}^{1/\alpha} &= \alpha^{-1}\int_0^\infty s^{(1-\alpha)/\alpha}\lambda(f,s)ds \\ &\leqq (\delta^{-1/\alpha}/\alpha)\int_0^\infty s^{(1-\alpha)/\alpha}\lambda(h,s)^{1/\alpha}ds \\ &= (\delta^{-1/\alpha}/\alpha)\int_0^\infty \lambda(h,s)[s\lambda(h,s)]^{(1-\alpha)/\alpha}ds \\ &\leqq (\delta^{-1/\alpha}/\alpha)\|h\|_1^{(1-\alpha)/\alpha}\int_0^\infty \lambda(h,s)ds \\ &= (\delta^{-1/\alpha}/\alpha)\|h\|_1^{1/\alpha}.\end{aligned}$$

That is,

$$\delta\alpha^\alpha\|f\|_{1/\alpha} \leqq \int_G \sup\{f(x-k): k\in K\}dm(x).$$

§2. Fix a positive integer $n \geqq 3$ and let $G(n)=\{0,1,\dots,n-1\}$. We will interpret $G(n)$ at times as a set of integers and at times as a realization of the group of integers modulo n, but the appropriate interpretation will be clear from the context. Fix a subset $S \subseteq G(n)$ such that $0 \in S$ and consider the generalized Cantor set $K \subseteq [0,1]$ consisting of all sums $\Sigma_{j=1}^\infty s_j n^{-j}$ such that each $s_j \in S$. (Thus if $n=3$ and $S=\{0,2\}$, K is the Cantor middle-thirds set, which we will denote by C in the sequel.) The normalized counting measure on $G(n)$ will be denoted by m_n, while m will henceforth stand for Lebesgue measure on $[0,1)$. We take $[0,1)$ with addition modulo one as a model for $\mathbf{T}$ and regard the Cantor sets K as compact subsets of $\mathbf{T}$ by identifying 0 and 1 if necessary. We will prove the following theorem.

THEOREM 2. *Fix $\alpha \in (0,1)$. Suppose that for every subset $E \subseteq G(n)$ the following inequality holds (where $E+S$ is computed in the group $G(n)$):*

$$m_n(E)^\alpha \leqq m_n(E+S). \tag{3}$$

Then the inequality

$$m(E)^\alpha \leqq m(E+K) \tag{4}$$

holds for every Borel $E \subseteq \mathbf{T}$.

COROLLARY. *If E is a Borel subset of $\mathbf{T}$, then (2) holds.*

PROOF OF COROLLARY. Taking $\alpha = 1 - \log 2/\log 3$, $n = 3$, and $S = \{0,2\}$ in Theorem 2, we must check that (3) holds. If card$(E) = 2$ or 3, then the right hand side of (3) is 1, and so (3) is true: If card$(E) = 1$, then (3) is the equality

$$\left(\frac{1}{3}\right)^{1-\log 2/\log 3} = \frac{2}{3}.$$

We would like to prove Theorem 2 by iterating some elementary inequality. This general strategy has been successful in the past — see [2]. The obvious inequality to try is (3), but an attempt to do so shows that the inequality actually needed here is

$$\|f\|_{L^{1/\alpha}(G_n)} \leqq \int_{G(n)} \sup\{f(x-s) : s \in S\} dm_n(x) \tag{5}$$

for $f \geqq 0$ on $G(n)$. Now (3) and (5) are, respectively, (a) and (c) of Theorem 1 with $G = G(n)$, $K = S$, and $\delta = 1$. Unfortunately, if $\delta = 1$ in (a) then Theorem 1 yields only $\delta = \alpha^\alpha$ in (c). But repeated iteration requires $\delta = 1$ in (c). Thus the proof of Theorem 2 will be in two parts. The first will be a proof that (3) does imply (5) and the second will be the iteration of (5). (Inequality (5) will be proved by the iteration of (3), but an auxiliary group is involved.)

LEMMA. *Fix $\alpha \in (0,1)$. For $i = 1,2$ suppose that G_i is a locally compact abelian group with Haar measure m_i and that K_i is a compact subset of G_i satisfying*

$$m_i(E_i)^\alpha \leqq m_i(E_i + K_i)$$

for each compact $E_i \subset G_i$. Then if $\tilde{G} = G_1 \times G_2$, $\tilde{m} = m_1 \times m_2$, and $\tilde{K} = K_1 \times K_2$, we have

$$\tilde{m}(E)^\alpha \leqq \tilde{m}(E + \tilde{K})$$

for every compact $E \subseteq \tilde{G}$.

PROOF. For a subset E of $\tilde{G}$, we will write $\chi(E; x, y)$ for the value of the indicator function of E at the point (x, y). Now

$$\begin{aligned}\tilde{m}(E)^\alpha &= \left[\int_{G_1} m_2\{y : (x,y) \in E\} dm_1(x)\right]^\alpha \\ &\leqq \left[\int_{G_1} m_2(\{y : (x,y) \in E\} + K_2)^{1/\alpha} dm_1(x)\right]^\alpha \\ &= \left[\int_{G_1} \left[\int_{G_2} \chi(E + \{0\} \times K_2; x, y) dm_2(y)\right]^{1/\alpha} dm_1(x)\right]^\alpha\end{aligned}$$

$$\leqq \int_{G_2} \left[\int_{G_1} \chi(E + \{0\} \times K_2; x, y) dm_1(x) \right]^\alpha dm_2(y)$$

$$= \int_{G_2} m_1\{x : (x, y) \in E + \{0\} \times K_2\}^\alpha dm_2(y)$$

$$\leqq \int_{G_2} m_1(\{x : (x, y) \in E + \{0\} \times K_2\} + K_1) dm_2(y)$$

$$= \int_{G_2} m_1\{x : (x, y) \in E + \tilde{K}\} dm_2(y)$$

$$= \tilde{m}(E + \tilde{K}).$$

Proof of (5). For $J = 1, 2, \ldots$ define G^J and S^J to be, respectively, the J-fold Cartesian products $\Pi_{j=1}^J G(n)$ and $\Pi_{j=1}^J S$. Put $G^\infty = \Pi_{j=1}^\infty G(n)$ and $S^\infty = \Pi_{j=1}^\infty S$. Let m^J and m^∞ be the normalized Haar measures on G^J and G^∞. Then repeated applications of (3) and the lemma show that

$$m^J(E)^\alpha \leqq m^J(E + S^J)$$

for $J = 1, 2, \ldots$ and $E \subseteq G^J$. The Fubini–Jessen theorem then yields

$$m^\infty(E)^\alpha \leqq m^\infty(E + S^\infty)$$

for any compact $E \subseteq G^\infty$. It follows from Theorem 1 that there is some $\delta > 0$ such that for $p = 1/\alpha$

$$\delta \| g \|_{L^p(G^\infty)} \leqq \int_{G^\infty} \sup\{g(x - s) : s \in S^\infty\} dm^\infty(x) \tag{6}$$

holds for all nonnegative continuous g on G^∞. If (5) failed there would be a nonnegative f (not identically zero) on $G(n)$ such that

$$\int_{G(n)} \sup\{f(x - s) : s \in S\} dm_n(x) = (1 - \varepsilon) \| f \|_{L^p(G(n))}$$

for some $\varepsilon \in (0, 1)$. Then for J so large that $(1 - \varepsilon)^J < \delta$ the function g defined on G^∞ by

$$g(x_1, x_2, \ldots) = \prod_{j=1}^J f(x_j)$$

would violate (6).

Proof of Theorem 2 (completion). For $J = 1, 2, \ldots$ let $G_J = G(n^J) = \{0, 1, \ldots, n^J - 1\}$. For the purposes of this proof write m_J for normalized counting measure on G_J. Let

$$S_J = \left\{ \sum_{j=0}^{J-1} s_j n^j : s_j \in S \right\} \subseteq G_J.$$

Writing p for $1/\alpha$ we will begin by using induction on J to establish the following inequality:

$$\|f\|_{L^p(G_J)} \leqq \int_{G_J} \sup\{f(x-s) : s \in S_J\} dm_J(x) \tag{7}$$

for $f \geqq 0$ on G_J. (The values of expressions like $x - s$ appearing as arguments for functions defined on a group — here G_J — are to be computed in the group.) For $J = 1$ inequality (7) is just (5). So assume that (7) holds with J replaced by $J - 1$ and fix a nonnegative function f on G_J. For $j = 0, 1, \ldots, n-1$ define $\tilde{f}_j$ on G_{J-1} by $\tilde{f}_j(x) = f(j + nx)$. For an integer x let $]x[$ denote the $j \in \{0, 1, \ldots, n-1\}$ such that $j \equiv x \pmod{n}$. Then

$$\|f\|_{L^p(G_J)} = \left(\frac{1}{n} \sum_{j=0}^{n-1} \|\tilde{f}_j\|^p_{L^p(G_{J-1})} \right)^{1/p} \leqq \frac{1}{n} \sum_{j=0}^{n-1} \sup\{\|\tilde{f}_{]j-s[}\|_{L^p(G_{J-1})} : s \in S\}$$

by (5). By (7) for $J - 1$ this last expression is not greater than

$$\frac{1}{n} \sum_{j=0}^{n-1} \left\| \sup_{s \in S} \tilde{f}_{]j-s[} \right\|_{L^p(G_{J-1})} \leqq \frac{1}{n} \sum_{j=0}^{n-1} \int_{G_{J-1}} \sup\{\tilde{f}_{]j-s[}(x - \tilde{s}) : s \in S, \tilde{s} \in S_{J-1}\} dm_{J-1}(x).$$

Since

$$\tilde{f}_{]j-s[}(x - \tilde{s}) = f(j + nx - (s + n\tilde{s})),$$

the last sum is the right hand side of (7). Next let $H_J = n^{-J} G_J$, let $K_J = n^{-J} S_J$, and regard H_J as a subgroup of $\mathbf{T}$. We will now use m_J to denote normalized counting measure on H_J. Then

$$K_1 \subseteq K_2 \subseteq \cdots \subseteq K$$

and K is the closure of $\bigcup_{J=1}^{\infty} K_J$ in $\mathbf{T}$. If $f \geqq 0$ is a continuous function on $\mathbf{T}$, then

$$\|f\|_{L^p(H_J)} \xrightarrow{J} \|f\|_{L^p(\mathbf{T})}.$$

Since

$$\sup\{f(x-k) : k \in K_J\} \xrightarrow{J} \sup\{f(x-k) : k \in K\}$$

uniformly in x, we also have

$$\int_{H_J} \sup\{f(x-k) : k \in K_J\} dm_J(x) \xrightarrow{J} \int_{\mathbf{T}} \sup\{f(x-k) : k \in K\} dm(x).$$

Thus (7) implies

$$\|f\|_{L^p(\mathbf{T})} \leqq \int_{\mathbf{T}} \sup\{f(x-k): k \in K\} dm(x)$$

for nonnegative continuous f on $\mathbf{T}$. Now (4) follows as in the proof of Theorem 1.

§3. Let λ be the Cantor–Lebesgue measure on the Cantor middle-thirds set C and write γ for $\log 2/\log 3$. Brown and Moran [1] proved the following inequality:

$$\lambda(K)^{1/(2\gamma)}\lambda(E)^{1/(2\gamma)} \leqq m(E+K) \tag{8}$$

for every pair of Borel sets $E, K \subseteq C$. Their proof is based on a geometric result of Woodall [4] which is obtained by iteration of a certain elementary inequality. We will iterate the same inequality in a different way to obtain Theorem 3(a) below — a result which is related to (8) as (c) of Theorem 1 is related to (a) of Theorem 1. In Theorem 3(b) we will obtain an analogous result which implies that

$$\lambda(K)m(E)^{1-\gamma} \leqq m(E+K) \tag{9}$$

for Borel sets $K \subseteq C$, $E \subseteq \mathbf{T}$. (The equations

$$\gamma \cdot \frac{1}{2\gamma} + \gamma \cdot \frac{1}{2\gamma} = 1, \qquad \gamma \cdot 1 + 1 \cdot (1-\gamma) = 1$$

together with the facts that the dimension of λ is γ and the dimension of m is 1 hint that something more general may be true here.) To state Theorem 3 we need a definition.

Definition. If f and g are nonnegative functions on a group G, the function $f\#g$ is defined by

$$f\#g(x) = \sup\{f(y)g(x-y) : y \in G\}.$$

Recall that for $p \in [1,\infty)$, $\|f\|_p$ denotes the norm of a suitable function f in $L^p(m)$.

Theorem 3. (a) *If f and g are continuous nonnegative functions on C (regarded as functions on $\mathbf{T}$ supported on C), then*

$$\|f\|_{L^{2\gamma}(\lambda)}\|g\|_{L^{2\gamma}(\lambda)} \leqq \|f\#g\|_1. \tag{10}$$

(b) *If f is as in* (a) *and $g \in C_c^+$, then*

$$\|f\|_{L^1(\lambda)}\|g\|_{1/(1-\gamma)} \leqq \|f\#g\|_1. \tag{11}$$

REMARK. It does not seem possible to deduce an inequality like (10) (or (11)) from (8) (or (9)) in a manner similar to the proof that (a) implies (c) in Theorem 1.

PROOF OF (a). For $J = 1,2,\ldots$ let $G_J = G(3^J) = \{0,1,\ldots,3^J - 1\}$ and let m_J be normalized counting measure on G_J. Let $C_J = \{\sum_{j=0}^{J-1} c_j 3^j : c_j = 0,2\}$, and let λ_J be normalized counting measure on C_J. The proof is similar to the proof of Theorem 2 after (5) was established. In particular, the crucial point is proving, for $J = 1,2,\ldots$, the inequality

$$\|f\|_{L^{2\gamma}(\lambda_J)}\|g\|_{L^{2\gamma}(\lambda_J)} \leqq \int_{G_J} f\#g\,dm_J \tag{12}$$

for nonnegative f and g on G_J with support in C_J. Given (12) a limit argument (very like that in the conclusion of the proof of Theorem 2) yields (10). Thus only (12) will be proved here. For $J = 1$ (12) is equivalent to the Lemma from [4]. So assume that (12) holds for $J - 1$ and let f, g be nonnegative functions supported in C_J. For $j = 0,1,2$ let $\tilde{f}_j$ be defined on G_{J-1} by $\tilde{f}_j(x) = f(j + 3x)$. Then $\tilde{f}_0$ and $\tilde{f}_2$ are supported on C_{J-1} and $\tilde{f}_1$ is zero. For an integer x, let $]x[$ be the $j \in \{0,1,2\}$ for which $j \equiv x \pmod 3$. Then

$$\|f\|_{L^{2\gamma}(\lambda_J)}\|g\|_{L^{2\gamma}(\lambda_J)} = \left(\frac{1}{2}\sum_{j=0}^{1}\|\tilde{f}_{2j}\|^{2\gamma}_{L^{2\gamma}(C_{J-1})}\right)^{1/(2\gamma)}\left(\frac{1}{2}\sum_{j=0}^{1}\|\tilde{g}_{2j}\|^{2\gamma}_{L^{2\gamma}(C_{J-1})}\right)^{1/(2\gamma)}$$

$$\leqq \frac{1}{3}\sum_{x=0}^{2}\sup_{c\in C_1}\|\tilde{f}_c\|_{L^{2\gamma}(C_{J-1})}\|\tilde{g}_{]x-c[}\|_{L^{2\gamma}(C_{J-1})}$$

by (12) for $J = 1$. By (12) for $J - 1$ this is not greater than

$$\frac{1}{3}\sum_{x=0}^{2}\sup_{c\in C_1}\int_{G_{J-1}}\sup\{\tilde{f}_c(\tilde{c})\tilde{g}_{]x-c[}(\tilde{x}-\tilde{c}) : \tilde{c}\in C_{J-1}\}\,dm_{J-1}(\tilde{x})$$

$$\leqq \frac{1}{3}\sum_{x=0}^{2}\int_{G_{J-1}}\sup\{f(c+3\tilde{c})g(x+3\tilde{x}-(c+3\tilde{c})) : c\in C_1, \tilde{c}\in C_{J-1}\}\,dm_{J-1}(\tilde{x}).$$

The last term is the right hand side of (12).

PROOF OF (b). With notation as in the proof of (a) it is not surprising that the heart of the proof is the inequality (for $J = 1,2,\ldots$)

$$\|f\|_{L^{1}(\lambda_J)}\|g\|_{L^{1/(1-\gamma)}(m_J)} \leqq \int_{G_J} f\#g\,dm_J \tag{13}$$

for nonnegative f and g on G_J with f supported on C_J. The inductive step is quite similar to the analogous part of the proof of (12). Thus we will give only the proof of (13) for $J = 1$. Let $\beta = 1 - \gamma$. After cancelling the normalizing constants for the measures and then scaling so that the left hand side of (13) is 1, (13) for $J = 1$ is the inequality

$$1 \leqq \max\{a_0 t, a_1(1-t)\} + \max\{a_1 t, a_2(1-t)\} + \max\{a_2 t, a_0(1-t)\} \tag{14}$$

for $0 \leqq t \leqq 1$ and $a_i \geqq 0$ satisfying

$$a_0^{1/\beta} + a_1^{1/\beta} + a_2^{1/\beta} = 1.$$

By continuity it is enough to prove (14) when each a_i is strictly positive. The right hand side of (14) defines a function $h(t)$ on $[0,1]$ such that

$$h(0) = h(1) = a_0 + a_1 + a_2 \geqq 1.$$

Let $t_1 = a_1/(a_0 + a_1)$, $t_2 = a_2/(a_1 + a_2)$, and $t_3 = a_0/(a_0 + a_2)$. Then t_1 is defined by the equation $a_0 t_1 = a_1(1 - t_1)$, and t_2 and t_3 arise similarly. It is enough to show that $h(t_i) \geqq 1$ for $i = 1,2,3$. Now $h(t_1) \geqq 1$ follows from the inequality

$$\begin{aligned} &a_0 a_1 + \max\{a_0 a_2, a_1^2\} + \max\{a_0^2, a_1 a_2\} \geqq a_0 + a_1 \\ &\text{whenever } a_i \geqq 0 \text{ satisfy } a_0^{1/\beta} + a_1^{1/\beta} + a_2^{1/\beta} = 1, \end{aligned} \tag{15}$$

and $h(t_2) \geqq 1$, $h(t_3) \geqq 1$ also follow from (15) by permuting the a_i. Writing $b_i = a_i^{1/\beta}$ transforms (15) into

$$\begin{aligned} &(b_0 b_1)^\beta + [\max\{b_0 b_2, b_1^2\}]^\beta + [\max\{b_0^2, b_1 b_2\}]^\beta \geqq b_0^\beta + b_1^\beta \\ &\text{whenever } b_0 + b_1 + b_2 = 1, \qquad b_i \geqq 0. \end{aligned} \tag{16}$$

Parametrizing this inequality by $c = b_2$, $\delta(1-c) = b_1$, $(1-\delta)(1-c) = b_0$ and cancelling a factor of $(1-c)^\beta$ show that it is enough to prove that

$$\begin{aligned} &[\delta(1-\delta)(1-c)]^\beta + [\max\{(1-\delta)c, \delta^2(1-c)\}]^\beta + \\ &[\max\{(1-\delta)^2(1-c), \delta c\}]^\beta - (1-\delta)^\beta - \delta^\beta \geqq 0 \end{aligned} \tag{17}$$

for $0 \leqq c \leqq 1$, $0 \leqq \delta \leqq \frac{1}{2}$. (The interval $0 \leqq \delta \leqq \frac{1}{2}$ suffices because (16) is symmetric in b_0 and b_1.) The left hand side of (17) defines a function $f_\delta(c)$ on $[0,1]$. This function is differentiable on $(0,1)$ except at the points $c_1 = c_1(\delta)$ and $c_2 = c_2(\delta)$ defined by the equations

$$(1-\delta)c_1 = \delta^2(1-c_1), \qquad (1-\delta)^2(1-c_2) = \delta c_2.$$

Since $f_\delta(1) = 0$ and

$$\frac{d^2}{dc^2} f_\delta(c) \leqq 0 \qquad \text{on } (0,1) \sim \{c_1, c_2\},$$

it is enough to show that

$$(18)\qquad f_\delta(0) \geqq 0, \qquad 0 \leqq \delta \leqq \tfrac{1}{2},$$

$$(19)\qquad f_\delta(c_1),\quad f_\delta(c_2) \geqq 0, \qquad 0 \leqq \delta \leqq \tfrac{1}{2}.$$

To establish (18) put

$$g(t) = [t(1-t)]^\beta + t^{2\beta} + (1-t)^{2\beta} - (1-t)^\beta - t^\beta.$$

Then

$$g(0) = 0, \qquad g(\tfrac{1}{2}) = 3 \cdot 2^{-2\beta} - 2^{1-\beta} > 0.$$

Also

$$g''(t) \leqq \beta(1-\beta)[(1-t)^{\beta-2}(1-t^\beta) + t^{\beta-2}(1-(1-t)^\beta)] - 2\beta^2(1-t)^{\beta-1}t^{\beta-1}.$$

Using the inequalities

$$1 - t^\beta \leqq \beta(1-t)t^{\beta-1}, \qquad 1-(1-t)^\beta \leqq \beta t(1-t)^{\beta-1}$$

one sees that this last expression is nonpositive. Thus $g(t) \geqq 0$ for $0 \leqq t \leqq 1/2$. Since $f_\delta(0) = g(\delta)$, (18) is true.

We turn to (19). Now

$$c_1 = \delta^2/(1-\delta+\delta^2), \qquad c_2 = (1-\delta)^2/(1-\delta+\delta^2).$$

Since $0 \leqq \delta \leqq 1/2$, it follows that $0 \leqq c_1 \leqq c_2 \leqq 1$. Also

$$\delta^2(1-c) \leqq (1-\delta)c \qquad \text{if and only if } c_1 \leqq c,$$

$$\delta c \leqq (1-\delta)^2(1-c) \qquad \text{if and only if } c \leqq c_2.$$

Using these facts one sees that $f_\delta(c_1) = f_\delta(c_2)$ and that this number will be nonnegative if and only if

$$(20)\quad [\delta(1-\delta)^2]^\beta + [\delta^2(1-\delta)]^\beta + (1-\delta)^{3\beta} - (1-\delta+\delta^2)^\beta[(1-\delta)^\beta + \delta^\beta] \geqq 0.$$

Write $k(\delta)$ for the left hand side of (20). Then

$$k(0) = k(\tfrac{1}{2}) = 0.$$

Thus it is enough to check that $k''(t) \leqq 0$ for $0 < t < \frac{1}{2}$. Now $k''(t)$ is naturally a sum of several terms, and showing that $k''(t) \leqq 0$ is simply a matter of pairing off certain of these terms. This is elementary but tedious. Here are the details. We can write

$$k''(t)=\sum_{j=1}^{6} k_j(t)$$

where

$$\begin{aligned} k_1(t) &= \beta(\beta-1)t^{\beta-2}(1-t)^{2\beta}+(-4\beta^2t^{\beta-1})(1-t)^{2\beta-1}+2\beta(2\beta-1)(1-t)^{2\beta-2}t^{\beta} \\ &= k_1^1(t)+k_1^2(t)+k_1^3(t), \\ k_2(t) &= \beta(\beta-1)(1-t)^{\beta-2}t^{2\beta}+(-4\beta^2t^{2\beta-1})(1-t)^{\beta-1}+2\beta(2\beta-1)t^{2\beta-2}(1-t)^{\beta} \\ &= k_2^1(t)+k_2^2(t)+k_2^3(t), \end{aligned}$$

$$k_3(t)=3\beta(3\beta-1)(1-t)^{3\beta-2},$$

$$k_4(t)=-(1-t+t^2)^{\beta}[\beta(\beta-1)(1-t)^{\beta-2}+\beta(\beta-1)t^{\beta-2}],$$

$$k_5(t)=-2\beta(1-t+t^2)^{\beta-1}(2t-1)[-\beta(1-t)^{\beta-1}+\beta t^{\beta-1}],$$

$$k_6(t)=-[\beta(\beta-1)(1-t+t^2)^{\beta-2}(2t-1)^2+2\beta(1-t+t^2)^{\beta-1}][(1-t)^{\beta}+t^{\beta}].$$

Recall that $\beta=1-\gamma$. Then the first bracketed factor in $k_6(t)$ can be written

$$\beta(1-t+t^2)^{\beta-1}\left[2-\gamma\left(4-\frac{3}{t^2-t+1}\right)\right].$$

This is positive for $0\leqq t\leqq\frac{1}{2}$, and so $k_6(t)\leqq 0$ for those t. Of the remaining terms, the k_i^j ($i=1,2$; $j=1,2,3$) are negative while k_3, k_4, and k_5 are positive on the interval of interest. One can check that

$$k_3(t)\leqq|k_2^2(t)|$$

and that

$$k_5(t)\leqq 2\beta^2(1-t+t^2)^{\beta-1}(1-2t)t^{\beta-1}\leqq 2\beta^2(1-t)^{2\beta-2}(1-2t)t^{\beta-1}\leqq\tfrac{1}{2}|k_1^2(t)|$$

for $0\leqq t\leqq\frac{1}{2}$. Using the inequalities

$$(1-t+t^2)^{\beta}-t^{2\beta}\leqq\beta(1-t)t^{2\beta-2},\qquad(1-t+t^2)^{\beta}-(1-t)^{2\beta}\leqq\beta t(1-t)^{2\beta-2}$$

one sees that

$$k_4(t)+k_1^1(t)+k_2^1(t)\leqq\beta^2(1-\beta)[(1-t)^{\beta-1}t^{2\beta-2}+t^{\beta-1}(1-t)^{2\beta-2}]$$

on $[0,\frac{1}{2}]$. Now, for these values of t,

$$\beta^2(1-\beta)t^{\beta-1}(1-t)^{2\beta-2}\leqq\tfrac{1}{2}|k_1^2(t)|,\qquad\beta^2(1-\beta)t^{2\beta-2}(1-t)^{\beta-1}\leqq|k_2^3(t)|.$$

Thus $k''(t)\leqq 0$ for $0<t<\frac{1}{2}$ and the proof of Theorem 3 is complete.

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